A Short Survey of Biminimal Legendrian and Lagrangian Submanifolds

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Abstract

We summarize recent results on biminimal Legendrian and Lagrangian submanifolds in Sasakian and complex space forms.

Keywords: Biminimal submanifolds, Legendrian submanifolds, Lagrangian submanifolds

1. Introduction

A minimal submanifold is one of the main objects in Riemmanian geometry. As a natural extension of minimal submanifolds from the view point of variational calculus, Loubeau and Montaldo [22] introduced the notion of biminimal submanifolds, which are critical points of the bienergy with respect to all normal variations. Here, the bienergy is defined by the integration of squared norm of tension field. Thus, it provides a measure for the extent to which a submanifold fails to be minimal. Minimal submanifolds are biminimal, however, the converse is not true in general. In fact, many examples of non-minimal biminimal submanifolds have been constructed in [20], [23], [24] and [29].

On the other hand, Legendrian and Lagrangian submanifolds are the most fundamental objects in contact and symplectic geometry, respectively. The study of them from the Riemmanian geometric point of view was initiated in 1970's (cf. [3, 9]). In particular, minimal Legendrian and Lagrangian submanifolds have attracted considerable attention from both geometric and physical point of view (cf. [18, 33]). Some extensions of such submanifolds also have been studied widely (see, [7, 8, 13, 26], for example). In this paper, as other extension, biminimal Legendrian and Lagrangian submanifolds are surveyed. But, all known results might have not been included.

2. Biharmonic maps and biminimal submanifolds

Let $f:(M^m,g)\to(\tilde{M},\tilde{g})$ be a smooth map between two Riemannian manifolds. The *tension field* $\tau(f)$ of f is a section of the vector bundle $f^*T\tilde{M}$ defined by

$$\tau(f):=\mathrm{tr}(\nabla^f df)=\sum_{i=1}^{m}\{\nabla^f_{e_i}df(e_i)-df(\nabla_{e_i}e_i)\},$$

where ∇^f, ∇ and $\{e_i\}$ denote the induced connection, the connection of M^m and a local orthonor-

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mal basis of M^m respectively. If $\tau(f)=0$ on M^m, f is called *harmonic*. The bienergy $E_2(f)$ of f over compact domain $\Omega\subset M^m$ is defined by

$$E_2(f)=\int_\Omega |\tau(f)|^2 dv_g,$$

where dv_g dvg is the volume form of M^m (see, [15]). Then E_2 provides a measure for the extent to which f fails to be harmonic. If f is a critical point of E_2 over every compact domain, then f is called a *biharmonic map* (or 2-*harmonic map*). In [21], Jiang proved that f is biharmonic if and only if

$$\mathcal{J}_f(\tau(f))=0,$$

where the operator $\mathcal{J}_f$ is the *Jacobi operator* defined by

$$\mathcal{J}_f(V):=\bar{\Delta}_f V-\mathcal{R}_f(V),\ V\in\Gamma(f^*T\tilde{M}),$$

$$\bar{\Delta}_f:=-\sum_{i=1}^m(\nabla^f_{e_i}\nabla^f_{e_i}-\nabla^f_{\nabla_{e_i}e_i}),$$

$$\mathcal{R}_f(V):=\sum_{i=1}^m R^{\tilde{M}}(V,\ df(e_i))df(e_i),$$

where $R^{\tilde{M}}$ is the curvature tensor of $\tilde{M}$. If f is an isometric immersion, (2.1) is rewritten as

$$\bar{\Delta}_f H=\mathcal{R}_f H,$$

where H is the mean curvature vector field.

The first example of nonminimal biharmonic submanifolds, i.e. submanifolds such that the inclusion map is biharmonic was constructed by Jiang as follows:

$$S^p\left(\frac{1}{\sqrt{2}}\right)\times S^q\left(\frac{1}{\sqrt{2}}\right)\to S^{p+q+1}(1),$$

where $p\neq q$ ([21]). For the classification of biharmonic submanifolds in spheres, see [5], [6] and [4], for example.

Recently, Loubeau and Montaldo introduced the notion of biminimal isometric immersions as biharmonic isometric immersions.

Definition 1 ([22]) An isometric immersion $f:(M,g)\to(\tilde{M},\ \tilde{g})$ is called a *biminimal* if it is a critical point of the bienergy functional E_2 with respect to all *normal variation* with compact support. Here, a normal variation means a variation f_t through $f=f_0$ such that the variational vector field $V=df_t/dt\,|_{t=0}$ is normal to M. In this case, M is called a *biminimal submanifold*.

A submanifold is biminimal if and only if

$$\{\bar{\Delta}_f H-\mathcal{R}_f(H)\}^\perp=0,$$

where $\perp$ means the normal part ([22]). Clearly, minimal submanifolds and biharmonic submanifolds are biminimal.

3. Biminimal Legendrian submanifolds in Sasakian space forms

Denote Sasakian space forms of constant ϕ-sectional curvature ϵ by $N^{2n+1}(\epsilon)$. The curvature tensor $\bar{R}$ of $N^{2n+1}(\epsilon)$ is given by ([25])

$$\begin{aligned}\bar{R}(X, Y)Z=&\frac{\epsilon+3}{4}\{\langle Y, Z\rangle X-\langle Z, X\rangle Y\}+\frac{\epsilon-1}{4}\{\eta(X)\eta(Z)Y\\&-\eta(Y)\eta(Z)X+\langle X, Y\rangle\eta(Y)\xi-\langle Y, Z\rangle\eta(X)\xi\\&+\langle Z, \phi Y\rangle\phi X-\langle Z, \phi X\rangle\phi Y+2\langle X, \phi Y\rangle\phi Z\},\end{aligned}$$

where η is a contact 1-form, ϕ is an almost contact structure and ξ is Reeb vector field of $N^{2n+1}(\epsilon)$. Sasakian space forms have been classified by Tanno (see, [32]). Let M^m be a submanifold in $N^{2n+1}(\epsilon)$. If η restricted to M^m vanishes, then M^m is called an *integral submanifold*, in particular if $m=n$, it is called a *Legendrian submanifold*. Integral curves are often called *Legendre curves*.

In case of $m=n=1$, Inoguchi proved ([19] and [20]) that

Theorem 2 *A Legendre curve in a Sasakian space from $N^3(\epsilon)$ is biminimal if and only if its curvature κ satisfies $\kappa^2=\epsilon-1$.*

For biharmonic Legendre curves in Sasakian space forms of general dimension, see [16] and [17].

Remark 3 In Proposition 4.1 of [1], the author and his collaborators showed the necessary and sufficient conditions for non-geodesic Legendre curves in 3-dimensional (κ, μ)-manifolds to be biharmonic. But, this is not true if $\kappa\neq 1$, in general. We must add the assumption that $T'\parallel \phi T$, where T is a unit tangent vector field of Legendre curve. In fact, if $\kappa\neq 1$, T' is not parallel to ϕT, in general. Therefore, we can not always choose a Frenet field with (4.1) in [1]. Authors thank J. Inoguchi and J.-E. Lee for pointing out this error.

The author classified nonminimal biharmonic Legendrian surfaces in 5-dimensional Sasakian space forms as follows:

Theorem 4 ([27]) *Let M^2 be a nonminimal biharmonic Legendrian surface in 5-dimensional Sasakian space forms $N^5(\epsilon)$ of constant ϕ-sectional curvature ϵ. Then $\epsilon\geq\frac{-11+32\sqrt{2}}{41}$ and at each point $p\in M^2$ there exists a suitable local coordinate system $\{x, y\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following :*

(1) $g=dx^2+dy^2$,

(2)

$$h(\partial_x, \partial_x)=\frac{\epsilon-1}{\alpha}\phi\partial_x,$$
$$h(\partial_y, \partial_y)=\left(\alpha-\frac{\epsilon-1}{\alpha}\right)\phi\partial_x,$$
$$h(\partial_x, \partial_y)=\left(\alpha-\frac{\epsilon-1}{\alpha}\right)\phi\partial_y,$$

where $\partial_x=\frac{\partial}{\partial_x}$, $\partial_y=\frac{\partial}{\partial_y}$ *and*

$$\alpha=\begin{cases}\sqrt{\dfrac{13\epsilon-9\pm\sqrt{41\epsilon^2+22\epsilon-47}}{8}} & (\epsilon\neq 1),\\ 1 & (\epsilon=1).\end{cases}$$

Conversely, suppose that ϵ *is a constant satisfying* $\epsilon\geq\frac{-11+32\sqrt{2}}{41}$ *and let* g *be the metric tensor on a simply-connected domain* $V\subset\mathbf{R}^2$ *defined by* (*1*). *Then, up to rigid motions of* $N^5(\epsilon)$, *there exists a unique Legendrian immersion of* (V, g) *into* $N^5(\epsilon)$ *whose second fundamental form is given by* (2). *Moreover such an immersion is nonminimal biharmonic.*

In particular, in the case that the ambient space is the unit 5-sphere, we have a explicit representation of such an immersion as follows:

Corollary 5 ([27]) *Let* $f: M^2\to S^5(1)\subset\mathbf{C}^3$ *be a nonminimal biharmonic Legendrian immersion into the unit* 5*-sphere. Then the position vector* $f(x, y)$ *of* M^2 *in* $\mathbf{C}^3$ *is given by*

$$f(x, y)=\frac{1}{\sqrt{2}}(e^{ix}, ie^{-ix}\sin\sqrt{2}y, ie^{-ix}\cos\sqrt{2}y).$$

As to biminimal Legendrian surfaces in 5-dimensional Sasakian space forms, we obtain

Theorem 6 ([29]) *Biminimal Legendrian surfaces in Sasakian space forms are biharmonic.*

Theorem 4 and 6 give us a complete classification of nonminimal biminimal Legendrian surfaces in 5-dimensional Sasakian space forms.

In [21] Jiang obtained the second variation formula for the bienergy E_2. But in case that the ambient space is not locally symmetric, it is difficult to compute the formula. We remark that Sasakian space forms are not locally symmetric in general. The second variation formula of biharmonic Legendrian submanifolds in Sasakian space forms was obtained by the author as follows:

Theorem 7 ([30]) *Let* f *be a biharmonic Legendrian immersion from a compact* n*-dimensional manifold* M^n *into a Sasakian space form* $N^{2n+1}(\epsilon)$. *Let* $\{f_t\}$ *be a smooth variation of* f

such that $f_0=f$ and V be the corresponding variational vector field. Then we have

$$\frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0}E_2(f_t)=\int_{M^n}\langle I(V),\ V\rangle dv_g,$$

where

$$\begin{aligned}I(V)=&-\frac{\epsilon+3}{4}\{|\tau|^2V+2\mathrm{trace}\langle\nabla^f_\cdot\tau,\ V\rangle\cdot+2\mathrm{trace}\langle\tau,\nabla^f_\cdot V\rangle\cdot+\langle\tau,\ V\rangle\tau\\&-2\mathrm{trace}\langle\nabla^f_\cdot V,\cdot\rangle\tau-(\mathcal{J}_fV)^\top+n\mathcal{J}_fV\}\\&+\frac{\epsilon-1}{4}\{\eta(V)\mathrm{trace}(\eta(\nabla^f_\cdot\tau)\cdot)+|\tau|^2\eta(V)\xi+2\mathrm{trace}\langle\nabla^f_\cdot\tau,\phi\cdot\rangle\phi V\\&-2\mathrm{trace}\langle\nabla^f_\cdot\tau,\ \phi V\rangle\phi\cdot-4\phi(\nabla^f_{(\phi V)^\top}V)-2\langle V,\ \phi\tau\rangle\xi+\eta(V)\phi\tau\\&-4\phi(\nabla^f_{\phi\tau}V)+2\mathrm{trace}\langle\tau,\ \phi(\nabla^f_\cdot V)\rangle\phi\cdot-3\langle\tau,\ \phi V\rangle\phi\tau+2\mathrm{trace}\langle\nabla^f_\cdot V,\ \phi\cdot\rangle\phi\tau\\&+2n\eta(V)\phi\tau+2\eta(V)(\phi V)^\top+n\eta(\mathcal{J}_fV)\xi-3(\mathcal{J}_fV)^\perp\}+\bar{\Delta}_f\mathcal{J}_fV,\end{aligned}$$

where $\top$ (resp. $\perp$) means the tangent (resp. normal) part.

Put

$$F(X):=\langle h(X,X),\ \phi X\rangle$$

for a vector field X of M^n. Then $F(\phi\tau)$ is a globally defined function on M^n. In terms of $\|\tau\|$ and $F(\phi\tau)$, we give the sufficient conditions for nonminimal biharmonic Legendrian submanifolds to be unstable as follows:

Theorem 8 ([30]) *Let M^n be a compact nonminimal biharmonic Legendrian submanifolds in a Sasakian space form $N^{2n+1}(\epsilon)$. If*

$$\int_{M^n}\{(\epsilon+3)\|\tau\|^4-3(\epsilon-1)F(\phi\tau)\}dv_g>0,$$

then M^n is unstable.

Applying Theorem 8 and the classification of biharmonic Legendrian curves and surfaces ([27]), we get

Corollary 9 ([30]) *Compact nonminimal biharmonic Legendrian submanifolds of Sasakian space forms $N^{2n+1}(\epsilon)$ are unstable if $n\leq 2$.*

There is a special vector field along submanifolds in contact manifolds, i.e., Reeb vector field ξ. Thus, it is natural and interesting to consider variations $V\in\mathrm{Span}\{\xi\}:=\{a\xi\,|\,a\in C^\infty(M)\}$. We call such variations *R-variations*. If the second variation under any *R-variation* is non-negative, f or M^n is said to be *R-stable*. Otherwise it is said to be *R-unstable*. Then, we obtain

Theorem 10 ([30]) *Let M^n be a compact nonminimal biharmonic Legendrian submanifold in Sasakian space form $N^{2n+1}(\epsilon)$. Then M^n is R-stable if and only if $\lambda_1 \geq \frac{3\epsilon-19}{4}$, where λ_1 is the first non-zero eigenvalue of the Laplacian acting on $C^\infty(M^n)$.*

Corollary 11 ([30]) *Compact nonminimal biharmonic Legendrian submanifolds of Sasakian space forms $N^{2n+1}(\epsilon)$ with $\epsilon \leq \frac{19}{3}$ are R-stable.*

4. Biminimal Lagrangian submanifolds in complex space forms

Let $\tilde{M}^n_s(4\epsilon)$ be a complex space form of complex dimension n and complex index $s(\geq 0)$. Denote $\tilde{M}^n_0(4\epsilon)$ by $\tilde{M}^n(4\epsilon)$. The complex index is defined as the complex dimension of the largest complex negative definite vector subspace of the tangent space. The curvature tensor $\tilde{R}$ of $\tilde{M}^n_s(4\epsilon)$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = \epsilon\{&\langle X, Z\rangle X - \langle X, Z\rangle Y + \langle Z, JY\rangle JX \\ &- \langle Z, JX\rangle JY + 2\langle X, JY\rangle JZ\},\end{aligned}$$

where J is the almost complex structure of $\tilde{M}^n_s(4\epsilon)$. Barros and Romero [2] showed that locally any complex space form $\tilde{M}^n_s(4\epsilon)$ is isometric holomorphically to $\mathbf{C}^n_s$, $\mathbf{C}P^n_s(4\epsilon)$ or $\mathbf{C}H^n_s(4\epsilon)$ according as $\epsilon=0$, $\epsilon>0$ or $\epsilon<0$. A submanifold of $\tilde{M}^n_s(4\epsilon)$ is called *Lagrangian* if J interchanges the tangent and the normal spaces.

As to biharmonic Lagrangian surfaces in complex space forms with $s=1$, we have

Theorem 12 ([28]) (I) *Biharmonic Lagrangian surfaces of non-zero constant mean curvature in $\mathbf{C}P^2_1(4)$ are locally given by*

$$\begin{aligned}f(x, y) = \pi\Big(&\sqrt{\frac{c^2}{c^2+1}}e^{-\frac{i}{c}x}, \sqrt{\frac{1}{c^2+1}}e^{icx}\cosh\sqrt{c^2+1}y, \\ &\sqrt{\frac{1}{c^2+1}}e^{icx}\sinh\sqrt{c^2+1}y\Big),\end{aligned}$$

where $c=\frac{\sqrt{7\pm\sqrt{41}}}{2}$ and π is the Hopf fibration.

(II) *Biharmonic Lagrangian surfaces of non-zero constant mean curvature in $\mathbf{C}H^2_1(-4)$ are locally given by*

$$\begin{aligned}f(x, y) = \pi\Big(&\sqrt{\frac{1}{b^2+1}}e^{-iby}\sinh\sqrt{b^2+1}x, \sqrt{\frac{1}{b^2+1}}e^{-iby}\cosh\sqrt{b^2+1}x, \\ &\sqrt{\frac{b^2}{b^2+1}}e^{\frac{i}{b}y}\Big),\end{aligned}$$

where $b=\frac{\sqrt{7\pm\sqrt{41}}}{2}$.

A submanifold is called *marginally trapped* (or quasi-minimal) if the mean curvature vector field is light-like. We obtain

Theorem 13 ([28]) *Let M be a marginally trapped biharmonic Lagrangian surface in $\tilde{M}^2_1(4\epsilon)$. Then $\epsilon=0$, and there exists a coordinate system $\{x, y\}$ defined in a neighborhood U of $p \in M$ such that the metric tensor of U is given by $g=dx^2-dx^2$ and the second fundamental form is given by the following form :*

$$h(\partial_x, \partial_x)=aJ\partial_x+\frac{a-c}{2}J\partial_y,\ h(\partial_y, \partial_y)=-cJ\partial_x-\frac{a+3c}{2}J\partial_y,$$

$$h(\partial_x, \partial_y)=-\frac{a-c}{2}J\partial_x+cJ\partial_y, \tag{4.1}$$

$$c=-\frac{x+y}{8}(\partial_x-\partial_y)f(x-y)+g(x-y),\ a=f(x-y)-c, \tag{4.2}$$

where f and g are arbitrary functions.

Conversely, suppose that a and c are functions on a simply-connected domain $U \subset \mathbf{R}^2$ defined by (4.2). *Let $g=dx^2-dy^2$ be the metric tensor on U. Then, up to rigid motions of $\mathbf{C}^2_1$, there exists a unique Lagrangian immersion of (U, g) into $\mathbf{C}^2_1$ whose second fundamental form is given by* (4.1). *Moreover such a surface is marginally trapped biharmonic.*

In [12], Chen and Ishikawa obtained the classification of marginally trapped biharmonic surfaces in $\mathbf{C}^2_1$. But it was not complete. In fact, surfaces in Theorem 13 were not included in their classification.

Remark 14 After the proof of Theorem 13 was completed, the author found out that the position vectors of Lagrangian surfaces in Theorem 13 were obtained explicitly by Chen and Dillen as follows (see, (i.2) in Theorem 4.1 of [11]) :

$$L(s, t)=c_1 s e^{if(t)}+z(t),$$

where $f(t)$ is a real-valued function, c_1 is a light-like vector, and $z(t)$ is a null curve in $\mathbf{C}^2_1$ satisfying $\langle iz'(t), c_1 e^{if(t)}\rangle=0$ and $\langle z'(t), c_1 e^{if(t)}\rangle=-1$. Here, a curve z is called null if it satisfies $\langle z', z'\rangle=0$. We can easily see that L is biharmonic.

As to biharmonic Lagrangian surfaces in complex space forms with $s=0$, we have

Theorem 15 ([28]) *Let M^2 be a biharmonic Lagrangian surface of non-zero constant mean curvature in $\tilde{M}^2(4\epsilon)$. Then $\epsilon>0$ and biharmonic Lagrangian surfaces of non-zero constant mean curvature in $\mathbf{C}P^2(4)$ are locally given by*

$$f(x, y)=\pi\Big(\sqrt{\frac{c^2}{c^2+1}}e^{-\frac{i}{c}x},\ \sqrt{\frac{1}{c^2+1}}e^{icx}\cos\sqrt{c^2+1}y,$$

$$\sqrt{\frac{1}{c^2+1}}e^{icx}\sin\sqrt{c^2+1y}\Big), \tag{4.3}$$

where $c=\frac{\sqrt{7\pm\sqrt{41}}}{2}$.

Clifford torus is defined by $T^{n+1}:=\{|z_i|=a_i|\sum_{i=1}^{n+1}a_i^2=1\}\in S^{2n+1}(1)$. Then $\pi(T^{n+1})$ is a Lagrangian submanifold in $\mathbf{C}P^n(4)$. A surface (4.3) is an open part of $\pi(T^3)$ with $a_1=\frac{\sqrt{9\pm\sqrt{41}}}{20}$, $a_2=a_3=\frac{\sqrt{11\mp\sqrt{41}}}{40}$. Zhang showed

Theorem 16 ([34]) *$\pi(T^{n+1})$ is biharmonic in $\mathbf{C}P^n(4)$ if and only if*

$$a_i\Big(\sum\frac{1}{a_i}\Big)-\frac{1}{a_i^3}=2(n+3)\Big((n+1)a_i-\frac{1}{a_i}\Big).$$

As to biminimal Lagrangian surfaces in complex space forms with $s=0$, we obtain

Theorem 17 ([31]) *Let M^2 be a biminimal Lagrangian surface in 2-dimensional complex space forms $\tilde{M}^2(4\epsilon)$, where $\epsilon\in\{-1,0,1\}$. If the mean curvature is non-zero constant, then $\epsilon=1$, i.e., the ambient space is the complex projective space $\mathbf{C}P^2(4)$, and moreover M^2 is an open part of $\pi(T^3)$ with $a_1=\frac{\sqrt{9\pm\sqrt{41}}}{20}$, $a_2=a_3==\frac{\sqrt{11\mp\sqrt{41}}}{40}$, i.e., it is locally represented by* (4.3).

Finally, we give an example of biminimal Lagrangian surface of non-constant mean curvature. We need the following existence theorem ([14]).

Theorem 18 *Let (M^n, g) be an n-dimensional simply connected Riemannian manifold. Let σ be a symmetric bilinear TM^n-valued form on M^n satisfying*

(1) *$g(\sigma(X, Y), Z)$ is totally symmetric,*
(2) *$(\nabla\sigma)(X, Y, Z)=\nabla_X\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\alpha(X, \nabla_X Z)$ is totally symmetric,*
(3) *$R(X, Y)Z=\epsilon(g(Y, Z)X-g(X, Z)Y)+\sigma(\sigma(Y, Z), X)-\sigma(\sigma(X, Z), Y)$.*

Then there exists a Lagrangian isometric immersion $x:(M^n, g)\to\tilde{M}^n(4\epsilon)$ such that the second fundamental form h satisfies $h(X, Y)=J\sigma(X, Y)$.

We consider the following autonomous system:

$$a'(x)=\gamma, \tag{4.4}$$
$$\gamma'(x)=-\mu\gamma+a(a^2+2c^2+\mu^2-2ac-5\epsilon), \tag{4.5}$$
$$c'(x)=(a-3c)\mu, \tag{4.6}$$
$$\mu'(x)=2c^2-\mu^2-ac-\epsilon, \tag{4.7}$$

where $\epsilon \in \{-1, 0, 1\}$. We take initial conditions as $a(0)=\gamma(0)=1$, $c(0)=\mu(0)=2$. Then for sufficiently small number t, there exist nowhere zero and non-constant solutions $a(x)$, $\gamma(x)$, $c(x)$, $\mu(x)$ on $(-t, t)$. We define the metric tensor g on $(-t, t)\times\mathbf{R}$ by

$$g=dx^2+\left(\exp\int^x \mu(x)dx\right)^2 dy^2, \tag{4.8}$$

and a symmetric bilinear form σ by

$$\begin{aligned} &\sigma(E_1, E_1)=(a-c)E_1, \\ &\sigma(E_1, E_2)=cE_2, \\ &\sigma(E_2, E_2)=cE_1, \end{aligned} \tag{4.9}$$

where $E_1=\partial_x$, $E_2=\dfrac{1}{\exp\int^x \mu(x)dx}\partial_y$.

By (4.6)-(4.9), we find that g and σ satisfy (1)-(3) in Theorem 18. Therefore there exists a Lagrangian isometric immersion from $((-t, t)\times\mathbf{R}, g)$ to $\tilde{M}^2(4\epsilon)$ whose second fundamental form is given by $h=J\sigma$. From (4.4) and (4.5), we obtain that it is biminimal. Moreover, since $a(x)\neq 0$ and $\gamma(x)\neq 0$ on $(-t, t)$, the mean curvature is nowhere zero and non-constant.

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